Guts Round

Lexington High School

April 9, 2016

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7th Annual Lexington Math Tournament - Guts Round - Part 1

1. [**5**] Today, the date 4/9/16 has the property that it is written with three perfect squares in strictly increasing order. What is the next date with this property?

2. [**5**] What is the greatest integer less than 100 whose digit sum is equal to its greatest prime factor?

3. [**5**] In chess, a bishop can only move diagonally any number of squares. Find the number of possible squares a bishop starting in a corner of a 20×16 chessboard can visit in finitely many moves, including the square it stars on.

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7th Annual Lexington Math Tournament - Guts Round - Part 2

4. [**5**] What is the fifth smallest positive integer with at least 5 distinct prime divisors?

5. [**5**] Let $\tau(n)$ be the number of divisors of a positive integer n, including 1 and n. How many positive integers $n \le 1000$ are there such that $\tau(n) > 2$ and $\tau(\tau(n)) = 2$?

6. [**5**] How many distinct quadratic polynomials $P(x)$ with leading coefficient 1 exist whose roots are positive integers and whose coefficients sum to 2016?

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7th Annual Lexington Math Tournament - Guts Round - Part 3

7. [**6**] Find the largest prime factor of 112221.

8. [**6**] Find all ordered pairs of positive integers (a, b) such that $\frac{a^2b^2+1}{ab-1}$ is an integer.

9. [**6**] Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is a function such that $f(2x) = f(1+x) + f(1-x)$ for all integers x. Find the value of $f(2)f(0) + f(1)f(6)$.

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7th Annual Lexington Math Tournament - Guts Round - Part 4

10. [**6**] For any six points in the plane, what is the maximum number of isosceles triangles that have three of the points as vertices?

11. [**6**] Find the sum of all positive integers n such that $\sqrt{n+\sqrt{n-25}}$ is also a positive integer.

12. [**6**] Distinct positive real numbers are written at the vertices of a regular 2016-gon. On each diagonal and edge of the 2016-gon, the sum of the numbers at its endpoints is written. Find the minimum number of distinct numbers that are now written, including the ones at the vertices.

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7th Annual Lexington Math Tournament - Guts Round - Part 5

13. [**7**] A 2016×2016 chess board is cut into $k \geq 1$ rectangle(s) with positive integer sidelengths. Let p be the sum of the perimeters of all k rectangles. Additionally, let m and M be the minimum and maximum possible value of $\frac{p}{k}$, respectively. Determine the ordered pair (m, M).

14. [**7**] For nonnegative integers n, let $f(n)$ be the product of the digits of n. Compute

$$\sum_{i=1}^{1000} f(i).$$

15. [**7**] How many ordered pairs of positive integers (m, n) have the property that mn divides 2016?

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7th Annual Lexington Math Tournament - Guts Round - Part 6

16. [**7**] Let a, b, c be distinct integers such that $a+b+c=0$. Find the minimum possible positive value of $|a^3+b^3+c^3|$.

17. [**7**] Find the greatest positive integer k such that $11^k - 2^k$ is a perfect square.

18. [**7**] Find all ordered triples (a, b, c) with $a \leq b \leq c$ of nonnegative integers such that $2a+2b+2c = ab+bc+ca$.

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7th Annual Lexington Math Tournament - Guts Round - Part 7

19. [**8**] Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(f(n)) + f(n+1) = n+2$ for all positive integers n. Find $f(20) + f(16)$.

20. [**8**] Let $\triangle ABC$ be a triangle with area 10 and $BC = 10$. Find the minimum possible value of $AB \cdot AC$.

21. [**8**] Let $\triangle ABC$ be a triangle with sidelengths $AB = 19, BC = 24, CA = 23$. Let D be a point on minor arc $\widehat{BC}$ of the circumcircle of $\triangle ABC$ such that $DB = DC$. A circle with center D that passes through B and C interests AC again at a point $E \neq C$. Find the length of AE.

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7th Annual Lexington Math Tournament - Guts Round - Part 8

22. [**8**] Let $m = \frac{1}{2}\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}$, where there are 2014 square roots. Let $f_1(x) = 2x^2 - 1$ and let $f_n(x) = f_1(f_{n-1}(x))$. Find $f_{2015}(m)$.

23. [**8**] How many ordered triples of integers (a, b, c) are there such that $0 < c \le b \le a \le 2016$, and $a + b - c = 2016$?

24. [**8**] In cyclic quadrilateral $ABCD$, $\angle BAD = 120°, \angle ABC = 150°, CD = 8$ and the area of $ABCD$ is $6\sqrt{3}$. Find the perimeter of $ABCD$.

7th Annual Lexington Math Tournament - Guts Round - Part 9

25. [**9**] Define a sequence $\{a_n\}_{n\ge1}$ of positive real numbers by $a_1 = 2$ and $a_n^2 - 2a_n + 5 = 4a_{n-1}$ for $n \ge 2$. Suppose k is a positive real number such that $a_n < k$ for all positive integers n. Find the minimum possible value of k.

26. [**9**] Let $\triangle ABC$ be a triangle with $AB = 13, BC = 14$, and $CA = 15$. Suppose the incenter of $\triangle ABC$ is I and the incircle is tangent to BC and AB at D and E, respectively. Line ℓ passes through the midpoints of BD and BE and point X is on ℓ such that $AX \parallel BC$. Find XI.

27. [**9**] Let x, y, z be positive real numbers such that $xy + yz + zx = 20$ and $x^2yz + xy^2z + xyz^2 = 100$. Additionally, let $s = \max(xy, yz, xz)$ and $m = \min(x, y, z)$. If s is maximal, find m.

7th Annual Lexington Math Tournament - Guts Round - Part 10

28. [**11**] Let ω_1 be a circle with center O and radius 1 that is internally tangent to a circle ω_2 with radius 2 at T. Let R be a point on ω_1 and let N be the projection of R onto line TO. Suppose that O lies on segment NT and $\frac{RN}{NO} = \frac{4}{3}$. Additionally, let S be a point on ω_2 such that T, R, S are collinear. Tangents are drawn from S to ω_1 and touch ω_1 at P and Q. The tangent to ω_1 at R intersects PQ at Z. Find the area of triangle $\triangle ZRS$.

29. [**11**] Let m and n be positive integers such that $k = \frac{m^2+n^2}{mn-1}$ is also a positive integer. Find the sum of all possible values of k.

30. [**11**] Let $f_k(x) = k \cdot \min(x, 1-x)$. Find the maximum value of $k \le 2$ for which the equation $f_k(f_k(f_k(x))) = x$ has fewer than 8 solutions for x with $0 \le x \le 1$.

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7th Annual Lexington Math Tournament - Guts Round - Part 11

In the following problems, A is the answer to Problem 31, B is the answer to Problem 32, and C is the answer to Problem 33. For this set, you should find the values of A, B, and C and submit them as answers to problems 31, 32, and 33, respectively. Although these answers depend on each other, each problem will be scored separately.

31. [**13**] Find

$$A\cdot B\cdot C+\cfrac{1}{B+\cfrac{1}{C+\cfrac{1}{B+\cfrac{1}{\ddots}}}}.$$

32. [**13**] Let $D = 7\cdot B\cdot C$. An ant begins at the bottom of a unit circle. Every turn, the ant moves a distance of r units clockwise along the circle, where r is picked uniformly at random from the interval $\left[\frac{\pi}{2D}, \frac{\pi}{D}\right]$. Then, the entire unit circle is rotated $\frac{\pi}{4}$ radians counterclockwise. The ant wins the game if it doesn't get crushed between the circle and the x-axis for the first two turns. Find the probability that the ant wins the game.

33. [**13**] Let m and n be the two-digit numbers consisting of the products of the digits and the sum of the digits of the integer $2016\cdot B$, respectively. Find $\frac{n^2}{m^2-mn}$.

Warning: The next round is the final round and will consist of three estimation problems.

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7th Annual Lexington Math Tournament - Guts Round - Part 12

34. [**15**] There are five regular platonic solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. For each of these solids, define its *adjacency angle* to be the dihedral angle formed between two adjacent faces. Estimate the sum of the adjacency angles of all five solids, in degrees. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15-\tfrac{1}{2}|A-E|\right\rfloor\right).$$

35. [**15**] Estimate the value of

$$\log_{10}\left(\prod_{k|2016} k!\right),$$

where the product is taken over all positive divisors k of 2016. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, \left\lceil 15\cdot\min\left(\frac{E}{A}, 2-\frac{E}{A}\right)\right\rceil\right).$$

36. [**15**] Estimate the value of $\sqrt{2016}^{\sqrt[4]{2016}}$. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, \left\lceil 15\cdot\min\left(\frac{\ln E}{\ln A}, 2-\frac{\ln E}{\ln A}\right)\right\rceil\right).$$

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